Section 4.5

L'Hôpital's Rule and Indeterminate Forms

(1) Refresher: Determinate and Indeterminate Forms

(2) L'Hôpital's Rule

(3) Comparing Growth of Functions

The Form of a Limit

The **form** of a limit $\lim_{x \to c} \square$ is the expression resulting from substituting $x = c$ into $\square$.

The form of a limit is **not** the same as its value!
It is a **tool for inspecting** the limit.

$\lim_{x \to 0} x^{\arctan(x)}$: form 0^0 $\qquad$ $\lim_{x \to \infty} (1+x)^{\frac{1}{x}}$: form ∞^0

$\lim_{x \to 0} \cos(x)^{\frac{1}{x}}$: form 1^∞ $\qquad$ $\lim_{x \to 0^+} \ln(x)\sin(x)$: form $0 \cdot \infty$

Indeterminate Forms are called indeterminate because they represent limits which may or may not exist and may be equal to any value. The form itself does **not** indicate the value of the limit. There are 7 indeterminate forms.

$$\frac{0}{0} \qquad \pm\frac{\infty}{\infty} \qquad \pm 0 \cdot \infty$$

$$1^{\infty} \qquad 0^{0} \qquad \infty^{0}$$

$$\infty - \infty$$

Indeterminate Forms

Limits of form $\dfrac{0}{0}$:

$$\lim_{x\to 0} \frac{\sin(x)}{x} = 1 \qquad \lim_{x\to 0} \frac{\sin^2(x)}{x} = 0 \qquad \lim_{x\to 0} \frac{\sin(x)}{x^2} \text{ DNE}$$

Limits of form $\dfrac{\infty}{\infty}$:

$$\lim_{x\to \infty} \frac{x+1}{x} = 1 \qquad \lim_{x\to \infty} \frac{\ln(x)}{x} = 0 \qquad \lim_{x\to \infty} \frac{e^x}{x} = \infty$$

These two indeterminate forms are like "tugs-of-war" between the numerator and denominator. Which of the two grows faster?

L'Hôpital's Rule

L'Hôpital's Rule

If f and g are differentiable near $x = a$ and either

(i) $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, or

(ii) $\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$,

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

The rule applies equally for one-sided limits.

Example 1:

These limits are of form $0/0$. The steps marked $\overset{LHR}{=}$ use L'Hôpital's Rule.

$$(a)\ \lim_{x\to 0} \frac{\sin(x)}{x} \overset{LHR}{=} \lim_{x\to 0} \frac{\frac{d}{dx}(\sin(x))}{\frac{d}{dx}(x)} = \lim_{x\to 0} \frac{\cos(x)}{1} = \cos(0) = 1$$

$$(b)\ \lim_{x\to 0} \frac{\ln(x^2+1)}{x} \overset{LHR}{=} \lim_{x\to 0} \frac{\frac{2x}{x^2+1}}{1} = 0$$

$$(c)\ \lim_{x\to 2} \frac{x^4+2x-20}{x^3-8} \overset{LHR}{=} \lim_{x\to 2} \frac{4x^3+2}{3x^2} = \frac{17}{6}$$

Example 2:

These limits are of form $\frac{\infty}{\infty}$. The steps marked $\overset{LHR}{=}$ use L'Hôpital's Rule.

$$(a)\ \lim_{x\to\infty} \frac{3x-7}{6x+5} \overset{LHR}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}(3x-7)}{\frac{d}{dx}(6x+5)} = \lim_{x\to\infty} \frac{3}{6} = \frac{1}{2}$$

$$(b)\ \lim_{x\to\infty} \frac{\ln(x)}{x} \overset{LHR}{=} \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln(x))}{\frac{d}{dx}(x)} = \lim_{x\to\infty} \frac{1/x}{1} = 0$$

$$(c)\ \lim_{x\to 0^+} \frac{\ln(x)}{\ln(\sin(x))} \overset{LHR}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{\cos(x)}{\sin(x)}} = \lim_{x\to 0^+} \left(\frac{\sin(x)}{x}\right)\left(\frac{1}{\cos(x)}\right) = 1$$

Example 3:

Sometimes it is necessary to perform L'Hôpital's Rule multiple times.

$$(a)\ \lim_{x\to 0} \frac{e^x - x - 1}{\cos(x) - 1}\ (0/0) = \lim_{x\to 0} \frac{e^x - 1}{-\sin(x)}\ (0/0)$$

$$= \lim_{x\to 0} \frac{e^x}{-\cos(x)}\ (\text{determinate}) = \boxed{-1.}$$

$$(b)\ \lim_{x\to\infty} \frac{-3x^3 - 20x^2}{2x^3 + x + 7}\ (\infty/\infty) = \lim_{x\to\infty} \frac{-9x^2 - 40x}{6x^2 + 1}\ (\infty/\infty)$$

$$= \lim_{x\to\infty} \frac{-18x - 40}{12x}\ (\infty/\infty)$$

$$= \lim_{x\to\infty} \frac{-18}{12} = \boxed{-\frac{3}{2}.}$$

L'Hôpital's Rule: Warnings

Warning #1:
L'Hôpital's Rule only applies to the indeterminate forms $0/0$ and ∞/∞. Before applying L'Hôpital's Rule to a limit, verify that it is of one of those two forms.

Warning #2:
Don't confuse L'Hôpital's Rule with the Quotient Rule! They have completely different uses.

- L'Hôpital's Rule is for evaluating limits.
- The Quotient Rule is for evaluating derivatives.

The Form $\infty - \infty$

To evaluate limits with the form $\infty - \infty$, use algebra to rewrite the expression as a quotient, often in $0/0$ or ∞/∞ form. Often this means finding a common denominator.

Example 4: To evaluate $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{x} - \csc(x)\right)$, which has the form $\infty - \infty$, rewrite it:

$$\frac{1}{x} - \csc(x) \quad = \quad \frac{1}{x} - \frac{1}{\sin(x)} \quad = \quad \frac{\sin(x) - x}{x\sin(x)}$$

This is an $0/0$ form, so we can apply L'Hôpital's Rule:

$$\lim_{x\to 0^+}\frac{\sin(x)-x}{x\sin(x)} \overset{LHR}{=} \lim_{x\to 0^+}\frac{\cos(x)-1}{x\cos(x)+\sin(x)} \ (0/0)$$

$$\overset{LHR}{=} \lim_{x\to 0^+}\frac{-\sin(x)}{-x\sin(x)+2\cos(x)} = \boxed{0}$$

The Form $0\cdot\infty$

Limits with the form $0\cdot\infty$ can easily be converted to $0/0$ or ∞/∞ form.

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$, then

$$\lim_{x\to a} \underbrace{f(x)g(x)}_{0\cdot\infty\text{ form}} = \lim_{x\to a} \underbrace{\frac{f(x)}{\left(\frac{1}{g(x)}\right)}}_{0/0\text{ form}} = \lim_{x\to a} \underbrace{\frac{g(x)}{\left(\frac{1}{f(x)}\right)}}_{\infty/\infty\text{ form}}$$

Examples (5):

(a) $\displaystyle \lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \overset{LHR}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \boxed{0}$

(b) $\displaystyle \lim_{x\to \frac{\pi}{2}^-} \left(x-\frac{\pi}{2}\right)\tan(x) = \lim_{x\to \frac{\pi}{2}^-} \frac{x-\frac{\pi}{2}}{\cot(x)} \overset{LHR}{=} \lim_{x\to \frac{\pi}{2}^-} \frac{1}{-\csc^2(x)} = \boxed{-1}$

Indeterminate Forms Involving Exponents

For limits of the forms 1^∞, 0^0, or ∞^0, the key step is to rewrite

$$\lim_{x\to c} \blacksquare \;=\; \lim_{x\to c} e^{\ln(\blacksquare)} \;=\; \lim_{x\to c} \exp(\ln(\blacksquare)) \;=\; \exp\left(\lim_{x\to c} \ln(\blacksquare)\right).$$

- The second equality is valid by the Limit Laws (see §2.4) because e^x is continuous everywhere.
- The "exp" notation avoids humongous expressions in superscript.

This technique changes the limit into one of form $0\cdot\infty$:

$$\begin{array}{ccccccc}
1^\infty & \to & e^{\ln|1^\infty|} & \to & e^{\infty\cdot\ln(1)} & \searrow & \\
0^0 & \to & e^{\ln|0^0|} & \to & e^{0\cdot\ln(0)} & \to & e^{0\cdot\infty} \\
\infty^0 & \to & e^{\ln|\infty^0|} & \to & e^{0\cdot\ln(\infty)} & \nearrow &
\end{array}$$

Indeterminate Forms Involving Exponents

Example 6: $\lim_{x\to 0^+} x^x$ has the indeterminate form 0^0.

$$\begin{aligned}
\lim_{x\to 0^+} x^x &= \exp\left(\lim_{x\to 0^+} \ln(x^x)\right) \\
&= \exp\left(\lim_{x\to 0^+} x\ln(x)\right) \qquad (0\cdot\infty) \\
&= \exp\left(\lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}}\right) \qquad (\infty/\infty) \\
&\overset{LHR}{=} \exp\left(\lim_{x\to 0^+} \frac{1/x}{-1/x^2}\right) = \exp\left(\lim_{x\to 0^+} (-x)\right) \\
&= \exp(0) = \boxed{1.}
\end{aligned}$$

Indeterminate Forms Involving Exponents

Example 7: $\lim_{x\to 0^+}(-\ln(x))^x$ has the indeterminate form ∞^0.

$$\lim_{x\to 0^+}(-\ln(x))^x = \exp\left[\lim_{x\to 0^+}\ln((-\ln(x))^x)\right]$$

$$= \exp\left[\lim_{x\to 0^+} x\ln(-\ln(x))\right]$$

$$= \exp\left[\lim_{x\to 0^+}\frac{\ln(-\ln(x))}{1/x}\right] \qquad (\infty/\infty)$$

$$\overset{LHR}{=} \exp\left[\lim_{x\to 0^+}\frac{\frac{1}{-\ln(x)}\cdot\frac{-1}{x}}{\frac{-1}{x^2}}\right] = \exp\left[\lim_{x\to 0^+}\frac{-x}{\ln(x)}\right]$$

$$= \exp(0) = 1.$$

Indeterminate Forms Involving Exponents

Example 8: Let a be any real number, so that $\lim_{x\to 0}(1+ax)^{1/x}$ has the indeterminate form 1^{∞}.

$$\begin{aligned}
\lim_{x\to 0}(1+ax)^{1/x} &= \exp\left[\lim_{x\to 0^+} \ln\left((1+ax)^{1/x}\right)\right] \\
&= \exp\left[\lim_{x\to 0^+} \frac{\ln(1+ax)}{x}\right] \qquad (0/0) \\
&\stackrel{LHR}{=} \exp\left[\lim_{x\to 0^+} \frac{\frac{a}{1+ax}}{1}\right] \quad = \quad \boxed{e^a.}
\end{aligned}$$

In fact, this calculation can be used to define the number e:

$$e = \lim_{x\to 0}(1+x)^{1/x}.$$

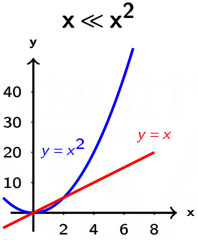
Comparing Growth Of Functions

We say that f grows faster than g if

(i) $\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)}=\infty$, or equivalently

(ii) $\displaystyle\lim_{x\to\infty}\frac{g(x)}{f(x)}=0.$

(Notation: $g \ll f$.)



Some important cases (that can all be checked using L'Hôpital's Rule):

(a) $x^n \ll e^x$ for all n.

(b) In fact, $x^n \ll a^x$ for all n and all $a>1$.

(c) $\log_a(x) \ll x^n$ for all n and all $a>0$.

Indeterminate Forms Involving Trigonometric Functions

Example 9: $\lim_{x\to 0} \dfrac{\tan(x)\sin(5x)}{x\sin(7x)}$ has the indeterminate form $\dfrac{0}{0}$.

$$\lim_{x\to 0} \frac{\tan(x)\sin(5x)}{x\sin(7x)} = \lim_{x\to 0}\left[\left(\frac{1}{\cos(x)}\right)\left(\frac{\sin(x)}{x}\right)\left(\frac{\sin(5x)}{\sin(7x)}\right)\right]$$

Limit of each fraction exists.

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$$= \lim_{x\to 0}\left(\frac{1}{\cos(x)}\right)\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)\lim_{x\to 0}\left(\frac{\sin(5x)}{\sin(7x)}\right)$$

$$\overset{LHR}{=} \lim_{x\to 0}\left(\frac{\cos(x)}{1}\right)\lim_{x\to 0}\left(\frac{5\cos(5x)}{7\cos(7x)}\right)$$

$$= \frac{5}{7}$$